

# SPACES GENERATED BY PRODUCTS OF EISENSTEIN SERIES

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**ABSTRACT.** We show, for levels of the form  $N = p^a q^b N'$  with  $N'$  squarefree, that in weights  $k \geq 4$  every cusp form  $f \in \mathcal{S}_k(N)$  is a linear combination of products of certain Eisenstein series of lower weight. In weight  $k = 2$  we show that the forms  $f$  which can be obtained in this way are precisely those in the subspace generated by eigenforms  $g$  with  $L(g, 1) \neq 0$ .

## 1. INTRODUCTION

The space  $\mathcal{M}_k(N, \chi)$  of modular forms of level  $\Gamma_0(N)$ , weight  $k$  and nebentypus  $\chi$  splits into the direct sum of the Eisenstein subspace  $\mathcal{E}_k(N, \chi)$  and the space of cusp forms  $\mathcal{S}_k(N, \chi)$ . It is straightforward to compute Fourier expansions and Hecke eigenforms in the Eisenstein subspace, but the space of cusp forms is far more mysterious, and any method of generating cusp forms is therefore of great interest. In this article we examine one of the simplest methods of generating cusp forms: What is the subspace of  $\mathcal{S}_k(N, \chi)$  generated by (the cuspidal projection of) products of Eisenstein series of lower weight?

For  $N = 1$  the answer to this question is very well-known: the graded ring<sup>1</sup>  $\bigoplus_{k \geq 0} \mathcal{M}_k(1)$  is a polynomial ring with two generators, one in degree four and one in degree six, corresponding to the Eisenstein series  $E_4$  and  $E_6$ . This means that every cusp form of level  $N = 1$  is a linear combination of products of Eisenstein series. However the number of products required to form the monomials in  $E_4$  and  $E_6$  for these linear combinations grows linearly with the weight  $k$ , which means these monomials are rather complicated. It is therefore natural to ask whether one can have simpler products at the expense of taking more Eisenstein series. Pushing this to the extreme we are led to ask: What is the subspace of  $\mathcal{S}_k(N, \chi)$  generated by (the cuspidal projection of) products of *two* Eisenstein series of lower weight?

Using the Rankin–Selberg method (as observed in [14] §5) one can show that, for  $k \geq 8$ ,  $\mathcal{M}_k(1)$  can be generated by the products  $E_l E_{k-l}$  for  $4 \leq l \leq k-4$ . Similar statements are known to hold for  $\mathcal{M}_k(p)$  for  $p$  prime and  $k \geq 4$  (see Imagoglu–Kohnen [4] for  $p = 2$ , Kohnen–Martin [5] for  $p > 2$ ). Recently Raum [10] has proven a different, rather general result: Let  $k \geq 12$  be an integer, let  $\rho$  be a representation of  $\mathrm{SL}_2(\mathbb{Z})$  on a complex vector space  $V$  such that  $\ker(\rho)$  contains a congruence subgroup, and define  $\mathcal{M}_k(\rho)$  to be the space of  $V$ -valued functions transforming as modular forms for the automorphy factor  $\gamma \mapsto (cz + d)^{-k} \rho(\gamma^{-1})$ . Then

$$\mathcal{M}_k(\rho) = \mathcal{E}_k(\rho) + \mathrm{span}_{\phi: \rho_M \otimes \rho_{M'} \rightarrow \rho} (T_M E_l \otimes T_{M'} E_{k-l}), \quad (1)$$

where  $4 \leq l \leq k-4$ ,  $\rho_M$  is the permutation representation on  $\Gamma_0(M) \backslash \mathrm{SL}_2(\mathbb{Z})$ , the  $E_k$  are corresponding vector-valued Eisenstein series, and the  $T_M$  are certain natural vector-valued Hecke operators. Finally, let us mention the work of Borisov–Gunnells [3], which uses the theory of toric varieties to define an interesting class of modular forms, called “toric modular forms”, which have simple Fourier expansions and yet generate the full space  $\mathcal{M}_k(N)$  for any  $N$ . This space of toric

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<sup>1</sup>When  $\chi = \mathbf{1}_N$  is the principal character modulo  $N$  we write  $\mathcal{M}_k(N)$  for  $\mathcal{M}_k(N, \mathbf{1}_N)$ .

modular forms contains certain (linear combinations of) Eisenstein series, and Borisov–Gunnells show that all cusp forms of sufficiently high weight are linear combinations of (the cuspidal projection of) products of two of these Eisenstein series, c.f. [3] Theorem 5.10.

In this paper we study a situation similar to Kohnen–Martin for higher level and non-trivial character. Thus we work with the space generated by products of the well-known Eisenstein series

$$E_l^{\phi,\psi}(z) = e_l^{\phi,\psi} + 2 \sum_{n \geq 1} \sigma_{l-1,\phi,\psi}(n) q^n \in \mathcal{M}_k(M, \phi\psi), \quad (2)$$

where  $\phi$  (resp.  $\psi$ ) is a primitive characters of level  $M_1$  (resp.  $M_2$ ),  $M_1 M_2 = M \mid N$ ,  $\sigma_{l-1,\phi,\psi}(n) = \sum_{d \mid n} \phi(n/d) \psi(d) d^{l-1}$ , and the constant term  $e_l^{\phi,\psi}$  (either zero or a value of a Dirichlet  $L$ -function) is uniquely determined by the modularity of  $E_l^{\phi,\psi}$ . We also use the images of these Eisenstein series under the lifting operator  $B_d$ , which acts on modular forms of weight  $k$  by  $f|B_d(z) = d^{\frac{k}{2}} f(dz)$ . In contrast to the set-ups of Raum and Borisov–Gunnells, our space of products is not *a priori* closed under the action of good Hecke operators, which complicates the matter. On the other hand, there are advantages to generating cusp forms directly from these standard Eisenstein series which we briefly discuss below.

In order to state our main result, when the character  $\chi$  is trivial, let us first define our space of generating products more precisely. Let  $k \geq 2$  be even, and fix a positive integer  $N$ . Let  $T$  be the set of primes  $p$  such that  $v_p(N) = 1$ , and write  $N_T = \prod_{p \in T} p$ . We then define  $\mathcal{Q}_k(N) \subset \mathcal{M}_k(N)$  to be the subspace generated by the products

$$E_l^{\phi,\psi} | B_{d_1} E_{k-l}^{\bar{\phi}, \bar{\psi}} | B_{d_2}$$

for all  $l \in \{1, \dots, k-1\}$ , all primitive characters  $\phi, \psi$  modulo all  $M_1, M_2$ , and all integers  $d_1, d_2$  such that the following conditions are satisfied:

$$\begin{aligned} \phi\psi(-1) &= (-1)^l, \\ (\phi, \psi, l) &\neq (1, 1, 2), (1, 1, k-2), \\ d_1 M_1 d_2 M_2 &\mid N, \\ d_1 M_1 &\mid N_T. \end{aligned} \quad (3)$$

The first three conditions are quite natural for defining a subspace of  $\mathcal{M}_k(N)$ . The fourth condition, and the overall shape of our products, is due to the fact that  $\mathcal{Q}_k(N)$  is closely related to a space  $\mathcal{P}_k(N)$  (defined in §4) formed by taking the image under the partial Atkin–Lehner operators of products of Eisenstein series supported at  $\infty$ . Our main result is:

**Theorem 1.1.** *Let  $k \geq 4$  be even. Let  $N = p^a q^b N'$  where  $a, b \in \mathbb{Z}_{\geq 0}$  and  $N'$  is squarefree, and let  $\mathcal{Q}_k(N) = \cup_{N_0 d \mid N} \mathcal{Q}_k(N_0) | B_d$  be the subspace of  $\mathcal{M}_k(N)$  generated by lifts of the spaces of products defined above. Then the restriction of the cuspidal projection to  $\mathcal{Q}_k(N)$  is surjective, i.e.*

$$\mathcal{M}_k(N) = \mathcal{Q}_k(N) + \mathcal{E}_k(N).$$

The case of weight 2 is different: Indeed, one sees immediately from the Rankin–Selberg method that the products of Eisenstein series are orthogonal to every newform  $f$  with vanishing central  $L$ -value, i.e.  $L(f, 1) = 0$ . Accordingly we define the space  $\mathcal{S}_{2, \text{rk}=0}(N)$  to be generated by newforms and lifts of newforms with non-zero central  $L$ -value. We obtain the analogue of Theorem 1.1 subject to this constraint.

**Theorem 1.2.** *Let  $N$  and  $\mathcal{Q}_2(N)$  be as in Theorem 1.1. Then*

$$\mathcal{S}_{2, \text{rk}=0}(N) \oplus \mathcal{E}_2(N) = \mathcal{Q}_2(N) + \mathcal{E}_2(N).$$

This phenomenon of isolating  $\mathcal{S}_{2,\text{rk}=0}(N)$  is also observed by Borisov–Gunnells [2].

We develop much of the theory to allow for more general level than  $N = p^a q^b N'$  and will discuss this restriction for the level  $N$  below. We also point out how similar results can be obtained when the character  $\chi$  is non-trivial by proving the analogue of Theorem 1.2 for  $\mathcal{S}_2(p, \chi)$ , see Theorem 4.5. Before we give an idea for the proof of the theorems we give a few explicit examples, and highlight some of the applications of such an expression for a newform:

- (1)  $N = 1, k = 12$ : The most well-known example is of course the discriminant modular form, which in our normalisation becomes

$$\Delta = \frac{50}{3} E_4^{1,1} E_8^{1,1} - \frac{147}{4} (E_6^{1,1})^2.$$

- (2)  $N = 11, k = 2$ : Let  $\phi$  be the character modulo 11 that maps 2 to  $\zeta_{10}$  and  $\psi$  the character that maps 2 to  $\zeta_{10}^3$ . Let  $f \in \mathcal{S}_2(11)$  be the unique newform in this space, then

$$f = \frac{1}{5} (-2\zeta_{10}^3 + 2\zeta_{10}^2 - \frac{1}{4}) E_1^{1,\phi} E_1^{1,\bar{\phi}} + \frac{1}{5} (2\zeta_{10}^3 - 2\zeta_{10}^2 - \frac{9}{4}) E_1^{1,\psi} E_1^{1,\bar{\psi}}.$$

- (3)  $N = 32, k = 2$ : Let  $\chi_4$  be the primitive character modulo 4 and  $\alpha$  the primitive character modulo 32 that maps 31 to 1 and 5 to  $\zeta_8$ . Let  $f \in \mathcal{S}_2(32, \chi_4)$  be the unique newform in this space, then

$$f = \frac{1}{8} (\zeta_8^3 - \zeta_8^2 + \zeta_8 - 1) E_1^{1,\chi_4\alpha} E_1^{1,\chi_4\bar{\alpha}} + \frac{1}{4} (\zeta_8^3 + \zeta_8^2) E_1^{1,\chi_4\alpha^2} \cdot E_1^{1,\chi_4\bar{\alpha}^2} |B_2.$$

Once established, an expression of a modular form  $f$  as a sum of products of Eisenstein series provides a way of calculating the Fourier expansion of  $f$  at  $\infty$ . Once such an expression for  $f$  is obtained,  $O(n \log(n))$  operations are required to compute  $n$  Fourier coefficients of  $f$  which is theoretically best possible,<sup>2</sup> and also appears to work well in practice.<sup>3</sup> There is, however, a cost: The most naive implementation involves computation the  $N$ th cyclotomic field, but it is plausible that computations in a much smaller field are sufficient.

Moreover, as mentioned in [10], one can use such expressions to compute Fourier expansions at *any* cusp of  $\Gamma_0(N)$ . The advantage of working directly with the Eisenstein series in (2) is that their Fourier expansions at cusps other than  $\infty$  are comparatively easy to obtain and were explicitly calculated by Weisinger [13]. This is in contrast to the set-up of Borisov–Gunnells, where it is not straightforward to obtain the expansion at other cusps from the representation of  $f$  as a sum of products (although we remark again that the generating “toric forms” of Borisov–Gunnells have a remarkably simple expansion at the cusp  $\infty$ ). When  $N$  is not squarefree the Fourier expansions at certain cusps are somewhat mysterious and difficult to access, yet carry important information. In future work we hope to use this method study the Fourier expansions at cusps of  $p$ -power denominator explicitly and their relationship to the finer arithmetic and representation-theoretic invariants of  $f$ .

Similarly, [13] also describes the action of the Atkin–Lehner operators on Eisenstein series, so once one has an explicit representation of a newform  $f$  as a linear combination of Eisenstein series it is straightforward to compute the Atkin–Lehner eigenvalues and the root number of  $f$ . This furnishes another example of an important datum which cannot immediately be read from the Fourier expansion of  $f$  at  $\infty$  when the level is not squarefree.

<sup>2</sup>We thank A. Booker for pointing this out.

<sup>3</sup>See <http://mathoverflow.net/q/221781/> for an example computed by D. Loeffler

Let us now give a sketch of the method to prove Theorem 1.1 (the proof of Theorem 1.2 requires minor modification). As hinted above, we argue for the most part with the space  $P_k(N)$ , and in §5 we show that  $P_k(N)$  has the same projection to  $\mathcal{S}_k^{\text{new}}(N)$  as  $Q_k(N)$ . If the projection of  $P_k(N)$  to  $\mathcal{S}_k^{\text{new}}(N)$  cuts out a proper subspace, we may pick  $f \in \mathcal{S}_k^{\text{new}}(N)$  orthogonal to  $P_k(N)$ . A standard application of the Rankin–Selberg method (§4) allows one to see that, if a *newform*  $f$  is orthogonal to  $P_k(N)$ , then all the critical  $L$ -values  $L(f_\psi | W_S^{NM}, j)$  must vanish (except for some cases when  $j = 2, k - 2$ , when the technical difficulties coming from weight two Eisenstein series enter). However, since our  $P_k(N)$  (or  $Q_k(N)$ ) might not be closed under the good Hecke operators, an element  $G$  of  $\mathcal{S}_k^{\text{new}}(N)$  orthogonal to  $P_k(N)$  will in general be a sum of newforms. On the other hand, our space  $P_k(N)$  will, by fiat, be closed under the action of the partial Atkin–Lehner operators  $W_\ell^N$  for  $\ell \in T$  (where  $T$  is the set of primes  $\ell$  with  $v_\ell(N) = 1$ ), and so we can at least assume that such a  $G$  is an eigenfunction of all these operators. Moreover we can modify  $G$  so that the orthogonality of  $G$  to  $P_k(N)$  is equivalent to the vanishing of many twisted  $L$ -values of  $G$ . We prove a general statement, possibly of independent interest, that if  $G$  is a cusp form which is an eigenfunction of certain Atkin–Lehner operators and for which sufficiently many twisted  $L$ -values vanish, then  $G = 0$ . The argument proceeds via modular symbols, and extends a result of Merel who proves a similar vanishing criterion in the case when  $G$  is a newform.

The reason the assumption  $N = p^a q^b N'$  enters is because we want to be in a situation where, if  $f$  is a newform (or a sum of newforms with the same  $W_\ell^N$ -eigenvalue for all  $\ell \in T$ ) and  $\alpha$  is a primitive character modulo  $M \mid N$ , then the  $W_S^{NM}$  (pseudo-)eigenvalues of  $f_\alpha$  for each set of prime divisors of  $N/M$  are determined by those of  $f$ . With our methods, this condition arises naturally in the proof of Theorem 1.1, and our argument would extend immediately to any situation where it holds. When  $N$  is squarefree this condition is automatic, since the twisting and Atkin–Lehner operators must commute (c.f. Proposition 2.3). When  $N$  is not squarefree this is a much more difficult question, and it seems unlikely that a purely local argument will work. Indeed our extension to level  $N = p^a q^b N'$  stems from a rather different argument using the modular symbols relations, which allows us to avoid this condition altogether when the number of prime factors of  $N$  is restricted as above.

Let us finish by remarking that one can easily compute (see §4) a trivial upper bound  $P$  for the dimension  $\dim \overline{Q_k(N)}$ , the projection  $\overline{Q_k(N)}$  of  $Q_k(N)$  to  $\mathcal{S}_k^{\text{new}}(N)$ . We compare this to the dimension  $\dim \mathcal{S}_k^{\text{new}}(N)$  in the “squarefree” and “prime-power” level aspects. In both cases the result is that  $P$  grows quicker than  $\dim \mathcal{S}_k(N)$ , although not by much, particularly in the prime-power case. When  $k = 2$  the problem of reducing the number of products required to generate  $\mathcal{S}_k^{\text{new}}(N)$ , or improving the bound on the upper bound  $P$  for the dimension by quantifying the amount of linear dependency amongst the products generating  $Q_k(N)$ , is perhaps interesting because of potential applications to the conjecture of Brumer on the number of newforms of level  $N$  for which  $L(f, 1) \neq 0$ .

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## 2. PRELIMINARIES

Let  $N, k \in \mathbb{Z}_{\geq 1}$  be a positive integers, and let  $\chi$  be a Dirichlet character modulo  $N$ . We keep the notations from the introduction for spaces of modular forms; we tacitly assume that  $\chi(-1) = (-1)^k$

since otherwise these spaces are zero. Our normalisation of the slash operator is

$$\left(f|_k \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right)(z) = \frac{(ad - bc)^{k/2}}{(cz + d)^k} f\left(\frac{az + b}{cz + d}\right),$$

so that scalars act trivially. We write  $|$  for  $|_k$  since the weight  $k$  will be clear from the context.

We write  $\mathbf{1}_N$  for the principal character modulo  $N$ , which satisfies  $\mathbf{1}_N(n) = 1$  for  $(n, N) = 1$ , and  $\mathbf{1}_N(n) = 0$  otherwise. We write  $\mathbf{1}$  for the trivial character, which satisfies  $\mathbf{1}(n) = 1$  for all  $n$ . If  $\chi$  is any character modulo  $N$  and  $S$  is a set of prime divisors of  $N$ , then we write  $\chi_S$  for the  $S$ -part of  $\chi$ . For a set  $S$  of prime divisors of  $N$  and a divisor  $M$  of  $N$ , we write  $M_S$  for the  $S$ -part of  $M$ , i.e.  $\prod_{p \in S} p^{v_p(M)}$ . We will also use the notation  $S_M = \{p \in S; p \mid M\}$  and  $\overline{S} = \{p \mid N; p \text{ prime}\} \setminus S$  (we will clarify the dependence on  $N$  when confusion may arise).

When  $(p, N) = 1$ , we write  $T_p$  for the Hecke operators on  $\mathcal{M}_k(N, \chi)$ ; these are extended multiplicatively to  $T_n$  for  $(n, N) = 1$ . When  $q \mid N$  we write  $U_q$  for the Hecke operators extended from the operators  $U_p$  (where  $p$  is a prime divisor of  $N$ ); the normalisation is

$$f|U_p = p^{k/2-1} \sum_{j=0}^{p-1} f| \begin{pmatrix} 1 & j \\ 0 & p \end{pmatrix}.$$

For a set of prime divisors  $S$  of  $N$  we define the Atkin–Lehner involution

$$W_S^N = \begin{pmatrix} N_S x & y \\ Nz & N_S w \end{pmatrix} \in \mathrm{M}_2(\mathbb{Z}),$$

where  $y \equiv 1 \pmod{N_S}$ ,  $x \equiv 1 \pmod{N_{\overline{S}}}$  and  $\det W_S^N = N_S$ . If  $M$  is a divisor of  $N$ , then we sometimes use the notation  $W_M^N$  for  $W_S^N$  with  $S = \{p \mid M\}$ . We simply write  $W_N$  for  $W_N^N = \begin{pmatrix} 0 & 1 \\ -N & 0 \end{pmatrix}$ . The following properties of  $W_S^N$  are well-known (see for example [1]):

**Proposition 2.1.** (i) Let  $S$  be a set of prime divisors of  $N$ . If

$$M = \begin{pmatrix} N_S x' & y' \\ Nz' & N_S w' \end{pmatrix}$$

is any matrix with  $x', y', z', w' \in \mathbb{Z}$  of determinant  $N_S$  then

$$f|M = \overline{\chi}_S(y') \overline{\chi}_{\overline{S}}(x') f|W_S^N. \quad (4)$$

In particular,  $W_S^N$  does not depend on the choice of  $x, y, z, w$ .

(ii) Let  $f \in \mathcal{M}_k(N, \chi)$ . Then

$$f|W_S^N \in \mathcal{M}_k(N, \overline{\chi}_S \chi_{\overline{S}}),$$

and  $W_S^N$  preserves the subspace of cusp forms.

(iii) If  $S$  and  $S'$  are disjoint sets of prime divisors of  $N$ , then

$$f|W_{S \cup S'}^N = \chi_{S'}(N_S) (f|W_S^N)|W_{S'}^N.$$

We also have

$$f|W_S^N|W_S^N = \chi_S(-1) \overline{\chi}_{\overline{S}}(N_S) f. \quad (5)$$

(iv) The adjoint of  $W_S^N$  on  $\mathcal{M}_k(N, \chi)$  with respect to the Petersson inner product is given by

$$W_S^{N,*} = \chi_S(-1) \chi_{\overline{S}}(N_S) W_S^N.$$

(v) Let  $p$  be a prime divisor of  $N$  with  $p \notin S$ . Then

$$f|U_p|W_S^N = \chi_S(p) f|W_S^N|U_p.$$

By a newform, we mean an element  $f \in \mathcal{S}_k(N, \chi)$  which is an eigenfunction of all Hecke operators, normalised to have first Fourier coefficient equal to one. We write  $\mathcal{S}_k^{\text{new}}(N, \chi)$  for the subspace of  $\mathcal{S}_k(N, \chi)$  generated by the newforms, so  $f \in \mathcal{S}_k^{\text{new}}(N, \chi)$  is a linear combination of newforms; we refer to these as elements of the new subspace.

Let  $q$  be a prime divisor of  $N$ . On the new subspace there is a close connection between the Hecke operator  $U_q$  and the Atkin-Lehner operator  $W_q^N$ :

**Proposition 2.2.** *Let  $\chi$  be a Dirichlet character modulo  $N$  and suppose  $\chi_q$  is principal. Let  $f$  be a newform of  $\mathcal{S}_k(N, \chi)$  with  $q$ -th Fourier coefficient  $a_q$  and Atkin-Lehner eigenvalue  $\lambda_q(f)$ .*

- If  $q^2 \mid N$  then  $a_q = 0$ .
- If  $q \mid N$  but  $q^2 \nmid N$  then  $\lambda_q(f) = -q^{1-\frac{k}{2}}a_q$  and hence we have the equality of operators

$$W_q^N = -q^{-\frac{k}{2}+1}U_q.$$

on  $\mathcal{S}_k^{\text{new}}(N, \chi)$ .

The third class of operators that play a major role for us are various twisting operators. Let  $f \in \mathcal{S}_k(N, \chi)$  with Fourier expansion  $f(z) = \sum_{n \geq 1} a_n e(nz)$ , let  $\alpha$  be a Dirichlet character modulo  $M$ , and define

$$f_\alpha(z) = \sum_{n \geq 1} a_n \alpha(n) e(nz).$$

With  $\alpha, f$  as above, define also

$$S_\alpha(f) = \sum_{a \bmod M} \overline{\alpha(a)} f|_k \begin{pmatrix} 1 & a/M \\ 0 & 1 \end{pmatrix}.$$

Note that if  $\alpha$  is primitive modulo  $M$  we have

$$S_\alpha(f) = G(\overline{\alpha}) f_\alpha. \tag{6}$$

For any  $z \in \mathfrak{H}$  we can view the function  $n' \mapsto \left( f|_k \begin{pmatrix} 1 & n'/N' \\ 0 & 1 \end{pmatrix} \right)(z)$  as a function  $F : (\mathbb{Z}/N'\mathbb{Z})^\times \rightarrow \mathbb{C}^\times$ , and we see by Fourier inversion that

$$f|_k \begin{pmatrix} 1 & n'/N' \\ 0 & 1 \end{pmatrix} = \sum_{\alpha \bmod N'} \frac{\alpha(n')}{\varphi(N')} S_\alpha(f), \tag{7}$$

the sum being over all Dirichlet characters modulo  $N'$ .

Finally we state some standard facts about the commutation relations for the above operators in the cases we will need them. These can be proved by direct computation (see also [1] §3).

**Proposition 2.3.** *Let  $N \in \mathbb{Z}_{\geq 1}$ , let  $f \in \mathcal{M}_k(N, \chi)$ , let  $\alpha$  be a Dirichlet character modulo  $N' \mid N$ . Then*

$$S_\alpha(f) \in \mathcal{M}_k(NN', \chi\alpha^2).$$

*Let  $q$  be any divisor of  $N$  that is coprime to  $N'$ , then*

$$S_\alpha(f)|U_q = \alpha(q) S_\alpha(f|U_q).$$

*Similarly, if  $S$  is a set of prime divisors of  $N$  such that  $N_S$  and  $N'$  are coprime, then*

$$S_\alpha(f)|W_S^{NN'} = \overline{\alpha}(N_S) S_\alpha(f|W_S^N).$$

### 3. A VANISHING CRITERION FOR CUSP FORMS

Before we prove the main theorem of this section we recall some facts from the theory of modular symbols; for details see [7] or [12] §8. Let  $k$  be an integer  $\geq 2$ . The space  $\mathbb{M}_k(\Gamma_1(N))$  of modular symbols is generated by the Manin symbols  $[P, g]$  where  $P$  is a homogeneous polynomial in  $\mathbb{C}[X, Y]$  of degree  $k-2$ , and  $g \in \mathrm{SL}_2(\mathbb{Z})$ . In fact the Manin symbols  $[P, g]$  only depends on  $P$  and the coset  $\Gamma_1(N)g$ . By mapping a matrix  $g$  to its bottom row modulo  $N$ , the cosets of  $\Gamma_1(N) \backslash \mathrm{SL}_2(\mathbb{Z})$  are in bijection with the set

$$E_N = \{(u, v) \in (\mathbb{Z}/N\mathbb{Z})^2; \gcd(u, v, N) = 1\};$$

see for example [12] Proposition 8.6. Note that  $\gcd(u, v, N)$  is well-defined, i.e. does not depend on the choice of representative of the residue classes  $u$  and  $v$ . We write  $[P, (u, v)] = [P, g]$  for any  $g \in \mathrm{SL}_2(\mathbb{Z})$  with bottom row congruent to  $(u, v)$  modulo  $N$ . For  $f \in \mathcal{S}_k(\Gamma_1(N))$  let  $\xi_f$  be the map

$$\xi_f([P, g]) = \int_{g0}^{g\infty} f(z)(gP)(z, 1)dz,$$

with  $g \in \mathrm{SL}_2(\mathbb{Z})$  acting on  $\mathbb{C}[X, Y]$  in the usual way.

Using the generators  $[X^j Y^{k-2-j}, (u, v)]$ , where  $0 \leq j \leq k-2$  and  $(u, v) \in E_N$ , we define  $\xi_f(j; u, v) = \xi_f([X^j Y^{k-2-j}, (u, v)])$ . Using the modular symbol relations, we obtain the relations

$$\xi_f(j; u, v) + (-1)^j \xi_f(k-2-j; v, -u) = 0, \quad (8)$$

$$\begin{aligned} \xi_f(j; u, v) + \sum_{i=0}^{k-2-j} (-1)^{k-2-i} \binom{k-2-j}{i} \xi_f(i; v, -u-v) \\ + \sum_{i=k-2-j}^{k-2} (-1)^i \binom{j}{i-k+2+j} \xi_f(i; -u-v, u) = 0, \end{aligned} \quad (9)$$

$$\xi_f(j; u, v) - (-1)^{k-2} \xi_f(j, -u, -v) = 0, \quad (10)$$

There is an involution  $\iota$  of  $\mathbb{M}_k(\Gamma_1(N))$ , namely  $\iota[X^j Y^{k-2-j}, (u, v)] = (-1)^{j+1} [X^j Y^{k-2-j}, (-u, v)]$ . Accordingly, we define

$$\xi_f^\pm(j; u, v) := \frac{\xi_f(j; u, v) \pm (-1)^{j+1} \xi_f(j; -u, v)}{2}.$$

The relations (8), (9), and (10) hold because of a relation on the underlying Manin symbols, see [12] §8.2.1. One can apply  $\iota$  to these relations for the Manin symbols to obtain another set of relations. Applying  $\xi_f$  and adding or subtracting as appropriate, we see that (8), (9), and (10) hold for  $\xi$  replaced by  $\xi_f^\pm$ .

By [7] Proposition 8 the maps  $f \mapsto \xi_f^+$  and  $f \mapsto \xi_f^-$  are injective, so  $f$  vanishes if all  $\xi_f^\pm(j; u, v)$  do. Note also that the  $\xi_f(j; u, v)$  are related to critical values of  $L$ -functions: Indeed, taking  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{SL}_2(\mathbb{Z})$  with  $(c, v) \equiv (u, v) \pmod{N}$  we have

$$\xi_f(j; u, v) = \frac{j!}{(-2\pi i)^{j+1}} L(f|g, j+1). \quad (11)$$

The main goal of this section is to prove Theorem 3.2, which is a vanishing criterion for  $f$  in terms of vanishing of certain twisted  $L$ -functions. The result is in the spirit of Corollaire 2 of [8], although we require some modifications since we do not assume that  $f$  is a newform, or even an eigenfunction of almost all Hecke operators. First we recall an identity from the proof of Proposition 6 in [8]:



**Lemma 3.1.** *Let  $N \in \mathbb{Z}_{\geq 1}$ , let  $(u, v) \in E_N$ , let  $S$  denote the set of prime divisors of  $N$  which divide  $u$ , let  $\overline{S}$  denote the remaining prime divisors of  $N$ , and let  $N'$  be the order of  $uv$  in  $\mathbb{Z}/N\mathbb{Z}$ . Let  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$  be such that  $(c, d) \equiv (u, v) \pmod{N}$ . Then*

$$\Gamma_1(N)g = \Gamma_1(N) \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{n}{N} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} NN'_S & 0 \\ 0 & N_S \end{pmatrix}^{-1},$$

where  $n$  is chosen so that  $n \equiv uv \pmod{N_{\overline{S}}}$  and  $n \equiv -uv \pmod{N_S}$ , and  $\begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \mathbb{Z}^{2 \times 2}$  has  $AD - BC = N_S N'_S$ ,  $A \equiv uN'_S \pmod{N_{\overline{S}}}$ ,  $B \equiv v/N_{\overline{S}} \pmod{N_S}$ , and  $N_S N'_S \mid A$ ,  $N_S N'_S \mid D$ ,  $NN' \mid C$ ,  $N_{\overline{S}} N'_S \mid B$ .

The existence of  $A, B, C, D$  follows from the Chinese Remainder Theorem. We omit the proof of this identity, which is simply a matter of checking that the matrix on the right hand side is integral with determinant one and with bottom row congruent to  $(u, v)$  modulo  $N$  (whence the conditions on  $A, B, C, D$ ).

Let  $N$  be a positive integer and let  $T$  be the set of prime divisors  $p$  of  $N$  with  $v_p(N) = 1$ , so

$$N = \prod_{p \in T} p \prod_{p \in \overline{T}} p^{v_p(N)},$$

where the exponents  $v_p(N)$  for  $p \in \overline{T}$  are all greater than 1.

**Theorem 3.2.** *Let  $N \in \mathbb{Z}_{\geq 1}$ ,  $k \geq 2$ , and let  $f \in \mathcal{S}_k^{\text{new}}(N)$  be an eigenfunction of all partial Atkin–Lehner operators  $W_p^N$  for  $p \in T$ , with  $T$  as above. Assume that  $L(f_\alpha | W_S^{NM}, j+1) = 0$  for all characters  $\alpha$  primitive modulo  $M \mid N$ , all  $j = 0, 1, \dots, k-2$  such that  $\alpha(-1) = (-1)^{j+1}$  (resp.  $\alpha(-1) = (-1)^j$ ), and all sets  $S \subseteq \overline{T}$  of prime divisors  $p$  that divide  $\frac{N}{M}$ . Then  $f = 0$ .*

*Proof.* We will present the argument for the case  $\alpha(-1) = (-1)^{j+1}$ , which uses the function  $\xi^+$ . The other case, using  $\xi^-$ , is almost identical, the only difference being which characters cancel in (14). We will show that the conditions in the theorem imply  $\xi_{f|W_N}^+(j; u, v) = 0$  for all  $j = 0, 1, \dots, k-2$  and  $(u, v) \in E_N$ , which in turn implies  $f = 0$ . Let us therefore fix  $(u, v) \in E_N$  and consider  $\xi_{f|W_N}^+(j; u, v)$ . As in the statement of Lemma 3.1, let  $S$  be the set of those prime divisors of  $N$  that divide  $u$  and write  $N'$  for the order of  $uv$  in  $\mathbb{Z}/N\mathbb{Z}$ . Note that every prime in  $S$  divides  $\frac{N}{N'}$ . Choose  $g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \text{SL}_2(\mathbb{Z})$  such that  $(c, d) \equiv (u, v) \pmod{N}$ . By Lemma 3.1 we have

$$\Gamma_1(N)g = \Gamma_1(N) \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \begin{pmatrix} 1 & \frac{n}{N} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} NN'_S & 0 \\ 0 & N_S \end{pmatrix}^{-1}, \quad (12)$$

with  $A, B, C, D$  and  $n$  satisfying the conditions of the lemma. Since  $f|W_N|W_N = f$ , we have

$$f|W_N|g = f| \begin{pmatrix} 1 & \frac{n}{N} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} NN'_S & 0 \\ 0 & N_S \end{pmatrix}^{-1}.$$

Now  $n \equiv uv \pmod{N_{\overline{S}}}$  and  $n \equiv -uv \pmod{N_S}$ , so  $n$  also has order  $N'$  modulo  $N$ . Hence  $nN' = n'N$  for some  $n'$  which is coprime to  $N'$ . Writing this as  $n/N = n'/N'$  and using (7) we get

$$f|W_N|g = \sum_{\alpha \bmod N'} \frac{\alpha(n')}{\phi(N')} S_\alpha(f) | \begin{pmatrix} A & B \\ C & D \end{pmatrix} \begin{pmatrix} NN'_S & 0 \\ 0 & N_S \end{pmatrix}^{-1},$$



where  $\alpha$  varies over all Dirichlet characters modulo  $N'$ . By Proposition 2.3 we have  $S_\alpha(f) \in \mathcal{S}_2(NN', \alpha^2)$ , and the conditions of Lemma 3.1 together with Proposition 2.1 give

$$S_\alpha(f) \left| \begin{pmatrix} A & B \\ C & D \end{pmatrix} \right. = \overline{\alpha_S^2}(B) \overline{\alpha_S^2} \left( \frac{A}{N_S N'_S} \right) S_\alpha(f) | W_S^{NN'}.$$

Hence, using (11),

$$\begin{aligned} \xi_{f|W_N}(j; u, v) &= \frac{j!}{(-2\pi i)^{j+1} \phi(N')} \sum_{\alpha} \alpha(n') \overline{\alpha_S^2}(B) \overline{\alpha_S^2} \left( \frac{A}{N_S N'_S} \right) L \left( S_\alpha(f) | W_S^{NN'} \left| \begin{pmatrix} \frac{1}{N N'_S} & 0 \\ 0 & \frac{1}{N_S} \end{pmatrix} \right. , j+1 \right) \\ &= \frac{j! (N_S / N'_S)^{\frac{k}{2}-j-1}}{(-2\pi i)^{j+1} \phi(N')} \sum_{\alpha} \alpha(n') \overline{\alpha_S^2}(B) \overline{\alpha_S^2} \left( \frac{A}{N_S N'_S} \right) L \left( S_\alpha(f) | W_S^{NN'} , j+1 \right), \end{aligned}$$

where the sum is over all characters modulo  $N'$ .

To compute  $\xi_{f|W_N}(j; -u, v)$  we proceed analogously with  $\tilde{g} = \begin{pmatrix} a & -b \\ -c & d \end{pmatrix}$ , since this has bottom row  $(-c, d) \equiv (-u, v) \pmod{N}$ . With  $A, B, C, D, n$  as in (12) we see that

$$\Gamma_1(N) \tilde{g} = \Gamma_1(N) \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix} \begin{pmatrix} 1 & -\frac{n}{N} \\ 0 & 1 \end{pmatrix} \begin{pmatrix} -A & B \\ C & -D \end{pmatrix} \begin{pmatrix} N N'_S & 0 \\ 0 & N_S \end{pmatrix}^{-1}. \quad (13)$$

The argument is as above, with  $n'$  replaced by  $-n'$ , and each individual summand in the final expression for  $\xi_{f|W_N}(j; u, v)$  changes by a factor of  $\alpha(-1) \overline{\alpha_S^2}(-1) = \alpha(-1)$ . From the definition of  $\xi_{f|W_N}^+$  we then see

$$\xi_{f|W_N}^+(j; u, v) = \frac{j! (N_S / N'_S)^{\frac{k}{2}-j-1}}{(-2\pi i)^{j+1} \phi(N')} \sum_{\alpha} \alpha(n') \overline{\alpha_S^2}(B) \overline{\alpha_S^2} \left( \frac{A}{N_S N'_S} \right) L \left( S_\alpha(f) | W_S^{NN'} , j+1 \right). \quad (14)$$

where the sum is over all characters  $\alpha$  modulo  $N'$  with  $\alpha(-1) = (-1)^{j+1}$ .

The next step is to relate  $S_\alpha(f)$  to the twist by the primitive character underlying  $\alpha$ . The key to this is the following lemma, which is proved by direct computation:

**Lemma 3.3.** *Let  $N$  and  $k$  be positive integers, let  $\chi$  be a Dirichlet character modulo  $N$ , and let  $f \in \mathcal{S}_k(N, \chi)$ . Let  $N' \in \mathbb{Z}_{\geq 1}$ , let  $\alpha$  be a character modulo  $N'$  with conductor  $M$ . Assume that  $M < N'$ , let  $p$  be any prime dividing  $N'/M$ , and let  $\beta$  be the character modulo  $N'/p$  inducing  $\alpha$ . Then*

$$S_\alpha(f) = p^{1-k/2} S_\beta(f|U_p) \left| \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right. - \overline{\beta}(p) S_\beta(f).$$

In our case  $f \in \mathcal{S}_k^{\text{new}}(N)$  is an eigenfunction of each  $W_p^N$  for  $p \in T$ , so by Proposition 2.2 it is also an eigenfunction of  $U_p$  for each  $p \in T$ . By the same proposition, if  $p \in \overline{T}$  then  $U_p$  is the zero operator, so  $f$  is trivially an eigenfunction of  $U_p$  in that case. In both cases we denote the  $U_p$ -eigenvalue of  $f$  by  $a_p$ . Then Lemma 3.3 gives

$$S_\alpha(f) = p^{1-k/2} a_p S_\beta(f) \left| \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \right. - \overline{\beta}(p) S_\beta(f),$$

and so

$$L(S_\alpha(f) | W_S^{NN'} , j+1) = (p^{-j} a_p - \overline{\beta}(p)) L(S_\beta(f) | W_S^{NN'} , j+1).$$

Applying this repeatedly we see that  $L(S_\alpha(f) | W_S^{NN'} , j+1)$  is a multiple of  $L(S_{\alpha_0}(f) | W_S^{NN'} , j+1)$ , where  $\alpha_0$  is the primitive character modulo  $M \mid N'$  inducing  $\alpha$  modulo  $N'$ . Next note

that  $S_{\alpha_0}(f) = G(\overline{\alpha_0})f_{\alpha_0} \in \mathcal{S}_k(NM, \alpha_0^2)$ . We then use  $S_{\alpha_0}(f)|W_S^{NN'} = S_{\alpha_0}(f)|W_S^{NM}|B_d$ , where  $d = (\frac{N'}{M})_S$ . Thus  $L(S_{\alpha_0}(f)|W_S^{NN'}, j+1)$  is a multiple of  $L(f_{\alpha_0}|W_S^{NM}, j+1)$ . If  $S \subseteq \overline{T}$ , then  $L(f_{\alpha_0}|W_S^{NM}, j+1) = 0$  by the conditions we pose on  $f$ .

Now suppose that there is a prime  $p \in S \cap T$ . Since  $v_p(N) = 1$  and  $p$  divides  $u$ , we have  $\gcd(p, N') = \gcd(p, M) = 1$ . Write  $S = S' \cup \{p\}$  and  $\alpha' = (\alpha_0)_{S'}$ . By Propositions 2.1 and 2.3 we have

$$f_{\alpha_0}|W_S^{NM} = \alpha'^2(p)(f_{\alpha_0}|W_p^{NM})|W_{S'}^{NM} = \alpha'^2(p)\overline{\alpha_0}(p)((f|W_p^{NM})_{\alpha_0})|W_{S'}^{NM}.$$

Since we assume that  $f$  is an eigenfunction of  $W_p^N$  we get that  $L(f_{\alpha_0}|W_S^{NM}, j+1)$  is a multiple of  $L(f_{\alpha_0}|W_{S'}^{NM}, j+1)$ . Applying this procedure for every prime  $p \in S \cap T$  we deduce that  $L(f_{\alpha}|W_S^{NM}, j+1)$  is a multiple of  $L(f_{\alpha_0}|W_{S''}^{NM}, j+1)$ , where  $S''$  is a set of prime divisors of  $N/M$  that is disjoint with  $T$ . Thus  $L(f_{\alpha}|W_S^{NM}, j+1) = 0$  by the conditions we pose on  $f$ .

Since  $\xi_{f|W_N}^+(j; u, v)$  is a sum of such  $L$ -values this shows that  $\xi_{f|W_N}^+(j; u, v) = 0$  for all  $(u, v) \in E_N$ , and hence  $f = 0$ . □

Theorem 3.2 is valid for all  $N \in \mathbb{Z}_{\geq 1}$ . Note that Atkin–Lehner operators are used in two different ways: first at  $p \in T$  where we insist  $f$  is an eigenfunction, second at  $S \subset \overline{T}$  where we insist the  $L$ -values of Atkin–Lehner images of twists vanish. For our applications we will restrict to  $N = p^a q^b N'$  with  $N'$  squarefree; a simple trick then allows us to do away with the latter use:

**Theorem 3.4.** *Let  $N = p^a q^b N'$  where  $p$  and  $q$  are distinct primes,  $a, b \in \mathbb{Z}_{\geq 0} \setminus \{1\}$ , and  $N'$  is squarefree and coprime to  $pq$ . Let  $k \geq 2$ , and let  $f \in \mathcal{S}_k^{new}(N)$  be an eigenfunction of all partial Atkin–Lehner operators  $W_{\ell}^N$  for primes  $\ell|N'$ . Assume that  $L(f_{\alpha}, j+1) = 0$  for all characters  $\alpha$  primitive modulo  $M|N$  and all  $j = 0, \dots, k-2$  with  $\alpha(-1) = (-1)^{j+1}$  (resp.  $\alpha(-1) = (-1)^j$ ). Then  $f = 0$ .*

*Proof.* We may assume that  $a > 1$  or  $b > 1$ , in particular that the set  $\overline{T} \subseteq \{p, q\}$  is non-empty, since otherwise this is just the statement of Theorem 3.2. Let  $(u, v) \in E_N$ . If no prime of  $\overline{T}$  divides  $u$  then the proof of Theorem 3.2 shows that  $\xi_{f|W_N}^+(j; u, v)$  is a linear combination of the  $L$ -values  $L(f_{\alpha}, j+1)$  so our assumptions give  $\xi_{f|W_N}^+(j; u, v) = 0$ . Similarly if no prime of  $\overline{T}$  divides  $v$  then  $\xi_{f|W_N}^+(k-2-j; v, -u) = 0$ ; we can then use the modular symbols relation (8) to see that  $\xi_{f|W_N}^+(j; u, v) = 0$ .

Now suppose that both  $u$  and  $v$  are divisible by a prime in  $\overline{T}$ . By the definition of  $E_N$ , it cannot be the case that the same prime divides both, so we are in the case when  $a > 1$  and  $b > 1$ , and we may assume that  $p$  divides  $u$  and  $q$  divides  $v$ . Then the residue class  $-u - v$  is divisible by neither  $p$  nor  $q$ , so  $\xi_f^+(i; v, -u - v) = \xi_f^+(i; -u - v, u) = 0$  for all  $0 \leq i \leq k-2$  by the above. Hence using (9) we obtain  $\xi_f(j; u, v) = 0$ . □

**Remark 3.5.** *More generally, a similar argument to Theorem 3.4 shows that one can restrict the sets  $S \subset \overline{T}$  in Theorem 3.2 to avoid two prescribed elements of  $\overline{T}$ .*

A technical difficulty arises in our application of Theorem 3.2 when  $k \geq 4$  due to the fact that the weight two Eisenstein series with trivial character is not holomorphic. To this end we prove a result which states that the problematic cases are in fact already a consequence of the other assumptions:

**Proposition 3.6.** *Let  $N \in \mathbb{Z}_{\geq 1}$ ,  $k \geq 4$  be even and  $f \in \mathcal{S}_k^{new}(N)$ . Assume that  $L(f_{\alpha}, j+1) = 0$  for all primitive characters  $\alpha$  modulo  $M|N$  where  $M > 1$ , and all  $j = 0, \dots, k-2$  with  $\alpha(-1) =$*

$(-1)^{j+1}$ . Assume moreover that  $L(f, j+1) = 0$  for all odd  $j \neq 1, k-3$ . Then  $L(f, 2) = 0$  and  $L(f, k-2) = 0$  must hold as well.

*Proof.* As in the proof of Theorem 3.4 we see  $\xi_{f|W_N}^+(j; u, v) = 0$  so long as  $j \neq 1, k-3$  and at least one of  $\gcd(u, N) = 1$  or  $\gcd(v, N) = 1$  holds. First assume  $k \geq 6$ . Applying (9) with  $j = 2$ ,  $(v, -u-v) = (0, 1)$ , using some of the vanishing we just observed, we get

$$-(k-4)\xi_{f|W_N}^+(1; 0, 1) - 2\xi_{f|W_N}^+(k-3; 1, 0) = 0.$$

Relation (8) with  $j = 1$ ,  $(u, v) = (0, 1)$  gives

$$\xi_{f|W_N}^+(1; 0, 1) - \xi_{f|W_N}^+(k-3; 1, 0) = 0,$$

hence

$$\xi_{f|W_N}^+(1; 0, 1) = 0,$$

since  $k \geq 6$ . Since  $\xi_{f|W_N}^+(1; 0, 1) = \xi_{f|W_N}(1; 0, 1)$ , (11) gives  $L(f|W_N, 2) = 0$ , hence  $L(f, k-2) = 0$ . The other case follows a similar argument, applying (9) with  $j = 2$  and  $(v, -u-v) = (1, 0)$  and (8) with  $j = 1$  and  $(u, v) = (1, 0)$  we get  $\xi_{f|W_N}^+(k-3; 0, 1) = 0$ , hence  $L(f, 2) = 0$ .

For  $k = 4$ , apply (9) with  $j = 0$ ,  $(v, -u-v) = (0, 1)$  to get

$$\xi_{f|W_N}^+(1; 0, 1) = \frac{1}{(-2\pi i)^2} L(f, 2) = 0.$$

□

#### 4. GENERATING SPACES OF CUSP FORMS BY PRODUCTS OF EISENSTEIN SERIES

We begin by recalling the theory of Eisenstein series as developed in [9] §7. Let  $N \in \mathbb{Z}_{\geq 1}$ ,  $l \in \mathbb{Z}_{\geq 1}$ , and let  $\phi$  (resp.  $\psi$ ) be Dirichlet characters modulo  $N_1$  (resp.  $N_2$ ) such that  $N_1 N_2 = N$ . We assume that  $\phi(-1)\psi(-1) = (-1)^l$ , and we extend  $\phi$  to a character of  $\Gamma_0(N)$  as usual by  $\phi\left(\begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) = \phi(d)$ , and similarly for  $\psi$ . Define the Eisenstein series

$$E_l^{\phi, \psi}(z, s) = \frac{(l-1)!N^l}{(-2\pi i)^l G(\overline{\psi}_0)} \sum_{(c, d) \in \mathbb{Z}^2 \setminus \{(0, 0)\}} \frac{\phi(c)\overline{\psi}(d)}{(Ncz + d)^l |Ncz + d|^{2s}},$$

which converges uniformly and absolutely for  $l + 2\operatorname{Re}(s) \geq 2 + \epsilon$ , for any  $\epsilon > 0$ . In the region of absolute convergence this satisfies the transformation law

$$E_l^{\phi, \psi}(\delta z, s) = \phi(\delta)\psi(\delta)j(\delta, z)^l |j(\delta, z)|^{2s} E_l^{\phi, \psi}(z, s) \quad (15)$$

for  $\delta \in \Gamma_0(N)$ . Since  $E_l^{\phi, \psi}(z, s)$  can be analytically continued in the  $s$ -variable, we can define  $E_l^{\phi, \psi}(z) = E_l^{\phi, \psi}(z, 0)$ . Moreover, unless  $l = 2$  and  $\psi$  is principal, the value at  $s = 0$  is a holomorphic function of  $z$ , so (15) along with some growth estimates shows that in fact  $E_l^{\phi, \psi} \in \mathcal{M}_l(N, \phi\psi)$ .

If  $\phi$  and  $\psi$  are primitive, the Fourier expansion of  $E_l^{\phi, \psi}$  can be deduced from Theorems 7.1.3, 7.2.12, and 7.2.13 of [9]:

$$E_l^{\phi, \psi}(z) = e_l^{\phi, \psi} + 2 \sum_{n \geq 1} \sigma_{l-1, \phi, \psi}(n) q^n \in \mathcal{M}_k(N, \phi\psi) \quad (16)$$

where  $\sigma_{l-1, \phi, \psi}(n) = \sum_{d|n} \phi(n/d)\psi(d)d^{l-1}$  and

$$e_l^{\phi, \psi} = \begin{cases} L(\psi, 1-l) & N_1 = 1, \\ L(\phi, 0) & N_2 = 1 \text{ and } l = 1, \\ 0 & \text{else.} \end{cases}$$

In the special case  $\phi = \mathbf{1}$  the Eisenstein series  $E_l^{\mathbf{1},\psi}(z, s)$ , appropriately normalised, is given by a Poincaré series:

$$\frac{2(-2\pi i)^l L(\bar{\psi}, l+2s) G(\bar{\psi}_0)}{(l-1)! N^l} E_l^{\mathbf{1},\psi}(z, s) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma_0(N)} \frac{\overline{\psi(\gamma)}}{j(\gamma, z)^l |j(\gamma, z)|^{2s}}. \quad (17)$$

Let  $k \in \mathbb{Z}_{\geq 1}$ ,  $\chi$  be a Dirichlet character modulo  $N$  with  $\chi(-1) = (-1)^k$ , and let  $f \in \mathcal{S}_k(N, \chi)$ . Given any  $g \in \mathcal{M}_l(N, \bar{\psi}\chi)$ , we consider the inner product

$$\langle g E_{k-l}^{\mathbf{1},\psi}(\cdot, s), f \rangle = \int_{\Gamma_0(N) \backslash \mathcal{H}} g(z) E_{k-l}^{\mathbf{1},\psi}(z, s) \overline{f(z)} y^{s+k} d\mu(z).$$

Here  $d\mu(z) = (dx dy)/y^2$  is the  $\mathrm{SL}_2(\mathbb{R})$ -invariant measure on the upper half plane. Note that integrand is indeed  $\Gamma_0(N)$ -invariant so the integral over this quotient makes sense, at least when it converges. This is certainly the case if  $s$  has sufficiently large real part, which we assume for the following proposition:

**Proposition 4.1.** *Let  $N, k, l \in \mathbb{Z}_{\geq 1}$ ,  $\chi$  be a Dirichlet character modulo  $N$ , and  $f$  be a newform in  $\mathcal{S}_k(N, \chi)$ , let  $\phi, \psi$  be Dirichlet characters such that  $\phi\psi = \chi$  and  $\phi(-1) = (-1)^l$ . Let  $\phi_0$  be the primitive character modulo  $M = \mathrm{cond}(\phi)$  associated to  $\phi$ . Exclude the two cases  $\phi_0 = \mathbf{1}$  and  $l = 2$ , and  $\phi = \chi$  and  $l = k - 2$ . Then*

$$\langle E_l^{\mathbf{1},\phi_0} E_{k-l}^{\mathbf{1},\psi}(\cdot, s), f \rangle = \frac{i^{k-l} \Gamma(s+k-1) (k-l-1)! N^{k-l}}{2^{2s+3k-l-2} \pi^{s+2k-l-1} L(\bar{\psi}, k-l+2s)^2 G(\bar{\psi}_0)} L(f^c, s+k-1) L((f^c)_{\phi_0}, s+k-l). \quad (18)$$

*Proof.* The Rankin-Selberg method (see [11]) shows

$$\langle g E_{k-l}^{\mathbf{1},\psi}(\cdot, s), f \rangle = \frac{\Gamma(s+k-1) (k-l-1)! N^{k-l}}{2(4\pi)^{s+k-1} (-2\pi i)^{k-l} L(\bar{\psi}, k-l+2s) G(\bar{\psi}_0)} \sum_{n \geq 1} \frac{\overline{a_n} b_n}{n^{s+k-1}}, \quad (19)$$

where  $a_n$  and  $b_n$  are the Fourier coefficients of  $f$  and  $g$ . Note that  $\overline{a_n}$  are the Fourier coefficients of  $f^c(z) = \overline{f(-\bar{z})} \in \mathcal{S}_k(N, \bar{\chi})$ . A standard computation (see e.g. [10] Proposition 4.1<sup>4</sup>) gives

$$\sum_{n \geq 1} \frac{\overline{a_n} \sigma_{l-1, \mathbf{1}, \phi_0}(n)}{n^{s+k-1}} = \frac{L(f^c, s+k-1) L((f^c)_{\phi_0}, s+k-l)}{L(\bar{\chi}\phi_0, 2s+k-l)}.$$

So taking  $g = E_l^{\mathbf{1},\phi}$  in (19) and using (16) we obtain the result.  $\square$

Note that both sides of (18) have analytic continuation, so by the uniqueness of analytic continuation the equality also holds at  $s = 0$ :

**Corollary 4.2.** *Under the hypotheses of Proposition 4.1,*

$$\langle E_l^{\mathbf{1},\phi_0} E_{k-l}^{\mathbf{1},\psi}, f \rangle = \frac{i^{k-l} (k-l-1)! (k-2)! N^{k-l}}{2^{3k-l-2} \pi^{2k-l-1} L(\bar{\psi}, k-l)^2 G(\bar{\psi}_0)} L(f^c, k-1) L((f^c)_{\phi_0}, k-l).$$

We can now proceed to the first result on generating cusp forms by products of Eisenstein series.

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<sup>4</sup>Our divisor function is  $\sigma_{l-1, \phi, \mathbf{1}}$  in Raum's notation.

**Definition 4.3.** Let  $N = p^a q^b N'$  be as in Theorem 3.4. For each  $M \mid N$ , write  $D(M)$  for the set of primitive Dirichlet characters modulo  $M$ . Let  $B(N) \subset \bigsqcup_{M \mid N} D(M) \times \{1, \dots, k-1\}$  consist of the pairs  $(\alpha, l)$  such that

$$\begin{aligned}\alpha(-1) &= (-1)^l, \\ (\alpha, l) &\neq (1, 2), (1, k-2).\end{aligned}$$

Define  $P_k(N) \subset \mathcal{M}_k(N)$  to be the space generated by the products

$$(E_l^{1,\alpha} E_{k-l}^{1,\overline{\alpha_N}}) | W_S^N$$

for all  $(\alpha, l) \in B(N)$  and all sets  $S$  of prime divisors of  $N'$ . Here  $\alpha_N$  denotes the extension of  $\alpha$  to a character modulo  $N$ .

**Theorem 4.4.** Let  $N = p^a q^b N'$  be as in Theorem 3.4. For  $M \subset \mathcal{M}_k(N)$ , write  $\overline{M}$  for the projection of  $M$  to  $\mathcal{S}_k^{\text{new}}(N)$ . Then for  $k \geq 4$  even

$$\overline{P_k(N)} = \mathcal{S}_k^{\text{new}}(N).$$

In the case  $k = 2$  we define  $\mathcal{S}_{2, rk=0}^{\text{new}}(N) \subset \mathcal{S}_2^{\text{new}}(N)$  to be the subspace generated by newforms  $f$  with  $L(f, 1) \neq 0$ . Then

$$\overline{P_2(N)} = \mathcal{S}_{2, rk=0}^{\text{new}}(N).$$

*Proof.* First assume  $k > 2$ . As in Section 3 denote the set of prime divisors of  $N'$  by  $T$ . Suppose that the containment  $\overline{P_k(N)} \subset \mathcal{S}_k^{\text{new}}(N)$  is proper. Since  $P_k(N)$  is closed under the action of the Atkin-Lehner operators  $W_S^N$  for  $S \subseteq T$ , so is the orthogonal complement of  $\overline{P_k(N)}$  in  $\mathcal{S}_k^{\text{new}}(N)$ . Therefore there exists a non-zero form  $g \in \mathcal{S}_k^{\text{new}}(N)$  that is orthogonal to  $P_k(N)$  and an eigenform of the  $W_S^N$ . We can write

$$g = \sum_{i=1}^r \beta_i f_i, \tag{20}$$

where  $f_1, \dots, f_r$  are the newforms in  $\mathcal{S}_k^{\text{new}}(N)$  with the same  $W_S^N$ -eigenvalues as  $g$  for all  $S \subseteq T$ . Using Corollary 4.2 (note  $f_i = f_i^c$  since the  $f_i$  have trivial character) we get

$$\langle E_l^{1,\alpha} E_{k-l}^{1,\overline{\alpha_N}}, f_i \rangle = \frac{(k-l-1)!(k-2)!N^{k-l}}{(-2\pi i)^{k-l}(4\pi)^{k-1}L(\alpha_N, k-l)L(\alpha, k-l)G(\alpha)} L(f_i, k-1)L((f_i)_\alpha, k-l).$$

Since  $g$  is a  $W_S^N$ -eigenform for  $S \subseteq T$  and the operators  $W_S^N$  are self-adjoint, for all  $S \subset T$  we have  $\langle E_l^{1,\alpha} E_{k-l}^{1,\overline{\alpha_N}} | W_S^N, g \rangle = 0$  if and only if  $\langle E_l^{1,\alpha} E_{k-l}^{1,\overline{\alpha_N}}, g \rangle = 0$ . We see that orthogonality of  $g$  to  $P_k(N)$  is equivalent to

$$\sum_{i=1}^r \beta_i L(f_i, k-1)L((f_i)_\alpha, k-l) = 0 \tag{21}$$

for all  $(\alpha, l) \in B(N)$ . Following an idea from the proof of Theorem 1 in [5], we define another form in  $G \in \mathcal{S}_k^{\text{new}}(N)$  by

$$G = \sum_{i=1}^r \beta_i L(f_i, k-1) f_i.$$

Since the  $f_i$  all have the same  $W_S^N$ -eigenvalues as  $g$  for  $S \subseteq T$ , so does  $G$ . Then (21) translates to

$$L(G_\alpha, k-l) = 0 \tag{22}$$

for  $(\alpha, l) \in B$ . Now applying Proposition 3.6 we see that  $L(G, 2) = 0$  and  $L(G, k-2) = 0$ . Thus  $G$  satisfies the conditions of Theorem 3.4, so  $G = 0$ . Since  $k \geq 4$ ,  $L(f_i, k-1) \neq 0$ , so all  $\beta_i$  must be

zero, and we arrive at the contradiction  $g = 0$ .

In the case where  $k = 2$  the proof is similar. The containment  $\overline{P_2(N)} \subset \mathcal{S}_{2,\text{rk}=0}^{\text{new}}(N)$  comes from Corollary 4.2, which shows that  $P_2(N)$  is orthogonal to every newform  $f$  with  $L(f, 1) = 0$ . The rest of the argument works as above.  $\square$

When  $a = b = 0$ , so  $N = N'$  is squarefree, one easily sees that the set  $B(N)$  in Definition 4.3 satisfies

$$\#B(N) \sim \frac{k-1}{2} \prod_{p|N} (p-1),$$

so, writing  $P$  for the number of generators for  $P_k(N)$ , we have

$$P \sim \frac{k-1}{2} \prod_{p|N} 2(p-1).$$

This should be compared to

$$\dim \mathcal{S}_k^{\text{new}}(N) \sim \frac{k-1}{12} \prod_{p|N} (p-1)$$

(c.f. [6]). It is therefore an interesting question whether we can remove the Atkin–Lehner operators appearing in the definition of  $P_k(N)$  so to obtain a space  $P_k(N)$  for which the number of generators is of similar size to the dimension of the target space. When  $N = p^a q^b$  we have

$$P \sim \frac{k-1}{2} p^{a-1} (p-1) q^{b-1} (q-1),$$

while (with  $a, b \geq 3$  for simplicity),

$$\dim \mathcal{S}_k^{\text{new}}(N) \sim \frac{k-1}{12} p^{a-1} (p-1) \left(1 - \frac{1}{p^2}\right) q^{b-1} (q-1) \left(1 - \frac{1}{q^2}\right).$$

In this case (as well as the case of  $N = N'$ ) it is also an interesting question whether one can quantify the linear dependency among the products of Eisenstein series generating  $P_k(N)$  (or  $Q_k(N)$ , defined below). These questions may be asked for any  $k$ , but they are particularly pertinent when  $k = 2$ , as any additional symmetry in this case which would allow one to lower the constant further could have application to Brumer’s conjecture regarding the number of newforms  $f$  with  $L(f, 1) = 0$ .

Most of the methods we have developed also work for the spaces  $\mathcal{M}_k(N, \chi)$  where  $\chi$  is a non-principal character modulo  $N$ . However there are some complications, in particular because the Atkin–Lehner operators  $W_S^N$  are no longer endomorphisms of  $\mathcal{M}_k(N, \chi)$  (when  $\chi$  is not quadratic) which means that one should take Eisenstein coming from different eigenspaces of the diamond operators to generate  $\mathcal{S}_k(N, \chi)$ . To minimise the technicalities we give an example of how the same methods can be used to treat the case of prime level, where the Atkin–Lehner operators are not necessary:

**Theorem 4.5.** *Let  $p$  be prime and let  $\chi$  a character modulo  $p$ . Let  $P_2(p, \chi)$  be the space generated by*

$$E_1^{1, \bar{\alpha}} E_1^{1, \bar{\chi}\alpha},$$

*for  $\alpha$  varying over all (primitive) odd characters modulo  $p$ . Write  $\overline{P_2(p, \chi)}$  for the projection of  $P_2(p, \chi)$  to  $\mathcal{S}_2(p, \chi)$ . Let  $\mathcal{S}_{2,\text{rk}=0}^{\text{new}}(p, \chi) \subset \mathcal{S}_2(p, \chi)$  be the subspace generated by newforms  $f$  with  $L(f, 1) \neq 0$ . Then*

$$\overline{P_2(p, \chi)} = \mathcal{S}_{2,\text{rk}=0}^{\text{new}}(p, \chi)$$

*Proof.* As above, Corollary 4.2 shows that the products  $E_1^{1,\bar{\alpha}} E_1^{1,\bar{\chi}\alpha}$  are orthogonal to any newform  $f$  with  $L(f, 1) = 0$  (since  $L(f, 1) = 0$  implies  $L(f^c, 1) = 0$ ), hence  $\overline{P_2(p, \chi)} \subseteq \mathcal{S}_{2, \text{rk}=0}^{\text{new}}(p, \chi)$ . Suppose for a contradiction that the reverse inclusion does not hold. As above we take a non-zero  $g \in \mathcal{S}_{2, \text{rk}=0}^{\text{new}}(p, \chi)$  that is orthogonal to  $\overline{P_2(p, \chi)}$  and write

$$g = \sum_{i=1}^r \beta_i f_i.$$

where the  $f_i$  are eigenforms that give a basis for  $\mathcal{S}_{2, \text{rk}=0}^{\text{new}}(p, \chi)$ . Orthogonality to  $P_2(p, \chi)$  translates to

$$\sum_{i=1}^r \beta_i L(f_i^c, 1) L((f^c)_\alpha, 1) = 0, \quad (23)$$

for all odd characters  $\alpha$  modulo  $p$ ; we again introduce

$$G = \sum_{i=1}^r \beta_i L(f_i^c, 1) f_i^c \in \mathcal{S}_2(p, \bar{\chi})$$

and note that (23) is equivalent to

$$L(G_\alpha, 1) = 0$$

for all odd characters  $\alpha$  modulo  $p$ . We will show that this implies  $\xi_{G|W_p}^+(0; u, v) = 0$  for all  $(u, v) \in E_p$  and hence  $G = 0$ . If  $p \mid u$ , then we may take  $u = 0$ , thus  $\xi_{G|W_N}^+(0; u, v) = \frac{1}{2}(\xi_{G|W_N}(0; 0, v) - \xi_{G|W_N}(0; 0, v)) = 0$ , so we can assume that  $p$  does not divide  $u$ . Using (8) with  $\xi$  replaced by  $\xi^+$  we get  $\xi_{G|W_N}^+(0; u, v) = 0$  when  $p \mid v$  as well, so we may assume  $p$  divides neither  $u$  nor  $v$ . Repeating the calculations in the proof of Theorem 3.2 in this case, so  $S = \emptyset$  and  $N' = p$ , we obtain

$$\xi_{G|W_p}^+(0; u, v) = \frac{1}{(-2\pi i)} \sum_{\alpha} \frac{\overline{\chi\alpha^2}(u)}{p-1} L(G_\alpha, 1) \quad (24)$$

where the sum is over all odd characters modulo  $p$  (c.f. (14)). We have  $L(G_\alpha, 1) = 0$  by orthogonality of  $g$ , so  $G = 0$ . Since  $L(f_i^c, 1) = \overline{L(f_i, 1)} \neq 0$  we see that  $\beta_i = 0$  for all  $i = 1, \dots, r$ , hence the contradiction  $g = 0$ .  $\square$

## 5. THE NEW PART OF $P_k(N)$

In this section we will analyse the new parts of the generators of  $P_k(N)$  for any  $N$ . We use this to construct another space  $Q_k(N)$  with the same projection to the new space as  $P_k(N)$  whose generators do not involve partial Atkin–Lehner operators. While  $P_k(N)$  was more useful for the proof of Theorem 4.4,  $Q_k(N)$  is more explicit and easy to implement on a computer. The first step is to write  $E_{k-l}^{1, \bar{\alpha}N}$  in terms of the Eisenstein series attached to the underlying primitive character:

**Lemma 5.1.** *Let  $\alpha$  be a primitive character modulo  $M$  with  $\alpha(-1) = (-1)^k$ . Writing  $N = \prod p^{v_p(N)}$ , let  $N_M = \prod_{p|M} p^{v_p(N)}$  be the  $M$ -part of  $N$ , so that  $M \mid N_M$  and  $\gcd(M, N/N_M) = 1$ . Then*

$$E_{k-l}^{1, \bar{\alpha}N} = \left(\frac{N}{M}\right)^{\frac{k-l}{2}} \sum_{e|N/N_M} \mu(e) \alpha(e) e^{-\frac{k}{2}+l} E_{k-l}^{1, \bar{\alpha}}|B_{N/M}e$$

*Proof.* For  $\text{Re}(s) \gg 0$  we have

$$E_{k-l, N}^{1, \bar{\alpha}N}(z, s) = \frac{(k-l-1)! N^{k-l}}{(-2\pi i)^{k-l} G(\bar{\alpha})} \sum_{(c, d) \neq (0, 0)} \frac{\alpha_N(d)}{(cNz + d)^{k-l} |cNz + d|^{2s}}.$$



Using the fact that  $\sum_{d|n} \mu(d)$  is the indicator function for  $n = 1$ , we get

$$\begin{aligned} \sum_{(c,d) \neq (0,0)} \frac{\alpha_N(d)}{(cNz + d)^{k-l} |cNz + d|^{2s}} &= \sum_{(c,d) \neq (0,0)} \sum_{e|\gcd(d, N/N_M)} \mu(e) \frac{\alpha(d)}{(cNz + d)^{k-l} |cNz + d|^{2s}} \\ &= \sum_{e|N/N_M} \frac{\mu(e)\alpha(e)}{e^{k-l+2s}} \sum_{(c,d) \neq (0,0)} \frac{\alpha(d)}{(cM(\frac{N}{Me})z + d)^{k-l} |cM(\frac{N}{Me})z + d|^{2s}} \\ &= \frac{(-2\pi i)^{k-l} G(\bar{\alpha})}{(k-l-1)! M^{k-l}} \sum_{e|N/N_M} \mu(e)\alpha(e) e^{-k+l-2s} E_{k-l}^{1, \bar{\alpha}}((N/Me)z, s). \end{aligned}$$

We obtain an equality of functions of the variable  $s$ , which remains true for  $s = 0$  by uniqueness of analytic continuation.  $\square$

Thus the product  $E_l^{1, \alpha} E_{k-l}^{1, \bar{\alpha}_N}$  is a linear combination of products of the form

$$E_l^{1, \alpha} \cdot \left( E_{k-l}^{1, \bar{\alpha}} |B_{N/Me} \right)$$

for  $e | N/N_M$ . If  $e \neq 1$  these products clearly have level smaller than  $N$ , so are old forms. Hence the projection of  $P_k(N)$  to the new space  $\overline{P_k(N)}$  is generated by the projections of the products

$$\left( E_l^{1, \alpha} |W_S^N \right) \cdot \left( E_{k-l}^{1, \bar{\alpha}} |B_{N/M} |W_S^N \right). \quad (25)$$

where  $S \subseteq T$  is a set of prime divisors of the squarefree part of  $N$ . Let us focus on the first factor for now. One easily sees that, as operators on  $\mathcal{M}_k(M, \chi)$ , we have the equality  $W_S^N = W_{S_M}^M |B_{(N/M)_S}$ , where  $S_M$  is the set of primes in  $S$  that divide  $M$ .

Using Proposition 14 of [13] we see that the first factor in (25) is a multiple of

$$E_l^{\bar{\alpha}_{S_M}, \alpha_{\overline{S_M}}} |B_{(N/M)_S},$$

where  $\overline{S_M} = \{p | M\} \setminus S_M$ . To study the second factor in (25) we use an extension of Proposition 1.5 of [1] that allows us to swap the order of the lifting operator and the Atkin–Lehner operator above:

**Lemma 5.2.** *Let  $F \in \mathcal{M}_k(M, \chi)$ ,  $d \in \mathbb{Z}_{\geq 1}$ , and  $S$  be a set of primes dividing  $dM$ . Let  $\overline{S} = \{p | dM\} \setminus S$ ,  $S_M = S \cap \{p | M\}$ , and define  $d_S = \prod_{p \in S} p^{v_p(d)}$  and  $d_{\overline{S}}$  as usual. Then*

$$F|B_d|W_S^{Md} = \overline{\chi}_S(d_{\overline{S}}) \overline{\chi}_{\overline{S}}(d_S) F|W_{S_M}^M |B_{d_{\overline{S}}}$$

*Proof.* Choose  $x, y, z, w \in \mathbb{Z}$  as in the definition of  $W_S^{Md}$ , i.e. satisfying  $y \equiv 1 \pmod{d_S M_S}$ ,  $x \equiv 1 \pmod{d_{\overline{S}} M_{\overline{S}}}$  and  $(M_S d_S)^2 xw - Mdzy = M_S d_S$ . As operators on  $\mathcal{M}_k(N, \chi)$ , we have

$$B_d W_S^{Md} = \begin{pmatrix} d & 0 \\ 0 & 1 \end{pmatrix} \begin{pmatrix} d_S M_S x & y \\ Mdz & d_S M_S w \end{pmatrix} = \begin{pmatrix} M_S d_S x & d_{\overline{S}} y \\ Mz & M_S w \end{pmatrix} \begin{pmatrix} d & 0 \\ 0 & d_S \end{pmatrix}.$$

The determinant of  $\begin{pmatrix} M_S d_S x & d_{\overline{S}} y \\ Mz & M_S w \end{pmatrix}$  is  $M_S$ , so by Proposition 2.1 and the fact that  $y \equiv 1 \pmod{M_S}$  and  $x \equiv 1 \pmod{M_{\overline{S}}}$  it equals  $\overline{\chi}_S(d_{\overline{S}}) \overline{\chi}_{\overline{S}}(d_S) W_{S_M}^M$ , whence the result.  $\square$

Applying Lemma 5.2 with  $d = N/M$  to  $E_{k-l}^{1, \bar{\alpha}} |B_{N/M} |W_S^N$  and using Proposition 14 of [13], we see that the second factor in (25) is a multiple of

$$E_{k-l}^{\alpha_{S_M}, \bar{\alpha}_{\overline{S_M}}} |B_{(\frac{N}{M})_{\overline{S}}},$$

so the product in (25) is a multiple of

$$\left( E_l^{\overline{\alpha}_{S_M}, \alpha_{\overline{S_M}}} | B_{(\frac{N}{M})_S} \right) \cdot \left( E_{k-l}^{\alpha_{S_M}, \overline{\alpha}_{\overline{S_M}}} | B_{(\frac{N}{M})_{\overline{S}}} \right).$$

In order to unwind this, set

$$\begin{aligned} M_1 = M_S &= \prod_{p \in S_M} p^{v_p(M)}, & M_2 = M_{\overline{S}} &= \prod_{p \in \overline{S_M}} p^{v_p(M)}, \\ d_1 = (N/M)_S &= \prod_{p \in S} p^{v_p(N) - v_p(M)}, & d_2 = (N/M)_{\overline{S}} &= \prod_{p \in \overline{S}} p^{v_p(N) - v_p(M)}. \end{aligned}$$

Note that  $\overline{S_M} = \{p \mid M\} \setminus S_M \subset \overline{S}$ . With these definitions,  $\overline{\alpha}_{S_M}$  (resp.  $\alpha_{\overline{S_M}}$ ) is a primitive character modulo  $M_1$  (resp.  $M_2$ ), which we now relabel as  $\phi$  (resp.  $\psi$ ). Note that as  $\alpha$  varies over all primitive characters of parity  $\epsilon$  modulo  $M$ ,  $\phi$  and  $\psi$  vary over all primitive characters modulo  $M_1$  and  $M_2$  such that  $\phi\psi$  has parity  $\epsilon$ . Now fix  $M \mid N$  and let  $S \subseteq T$  vary: we obtain all  $M_1, M_2$  such that  $M_1 \mid N_T$  and  $M_1 M_2 = M$ , and for given  $M_1, M_2$  we obtain all  $d_1, d_2$  such that  $d_1 M_1 \mid N_T$  and  $d_1 M_1 d_2 M_2 = N$ .

**Definition 5.3.** Let  $N \in \mathbb{Z}_{\geq 1}$ . Let  $T$  be the set of primes  $p$  such that  $v_p(N) = 1$ . Let  $B'(N)$  consist of all quintuples  $(\phi, \psi, l, d_1, d_2)$ , where  $\phi$  varies over all primitive characters of level  $M_1$ , with  $M_1$  varying over all divisors of  $N_T$ ,  $\psi$  varies over all primitive characters of level  $M_2$ , with  $M_2$  varying over all divisors of  $N$ , all  $l \in \{1, \dots, k-1\}$ , and all  $d_1, d_2 \in \mathbb{Z}_{\geq 1}$ , such that <sup>5</sup>

$$\begin{aligned} \phi\psi(-1) &= (-1)^l, \\ (\phi, \psi, l) &\neq (1, 1, 2), (1, 1, k-2), \\ d_1 M_1 &\mid N_T, \\ d_1 M_1 d_2 M_2 &= N. \end{aligned}$$

We define  $Q_k(N)$  to be the vector space generated by

$$E_l^{\phi, \psi} | B_{d_1} \cdot E_{k-l}^{\overline{\phi}, \overline{\psi}} | B_{d_2}. \quad (26)$$

for all  $(\phi, \psi, l, d_1, d_2) \in B'(N)$ .

The above calculation shows that  $Q_k(N)$  and  $P_k(N)$  have the same projection on to the new subspace  $\mathcal{S}_k^{\text{new}}(N)$ . Using the spaces  $Q_k(N)$  and their lifts we can extend Theorem 4.4 to the full space  $\mathcal{S}_k(N)$ :

**Theorem 5.4.** Let  $N$  be as in Theorem 4.4 and  $\mathcal{Q}_k(N) = \bigcup_{N_0 d \mid N} Q_k(N_0) | B_d$  be the subspace of  $\mathcal{M}_k(N)$  generated by the products

$$E_l^{\phi, \psi} | B_{d_1 d} \cdot E_{k-l}^{\overline{\phi}, \overline{\psi}} | B_{d_2 d}$$

for  $(\phi, \psi, l, d_1, d_2) \in B'(N_0)$  (as in Definition 5.3). Then for  $k \geq 4$

$$\mathcal{M}_k(N) = \mathcal{Q}_k(N) + \mathcal{E}_k(N).$$

*Proof.* This follows from Theorem 4.4, the previous calculations, and an inductive argument using the fact that

$$\mathcal{S}_k(N) = \bigcup_{N_0 \mid N} \bigcup_{d \mid N/N_0} \mathcal{S}_k^{\text{new}}(N_0) | B_d.$$

□

<sup>5</sup>Of course the second condition is only relevant when  $M = M_1 = M_2 = 1$ .

To treat the case  $k = 2$  we need one more result:

**Proposition 5.5.** *Let  $f \in \mathcal{S}_2^{\text{new}}(N_0)$  be a newform of level  $N_0 \mid N$  with  $L(f, 1) = 0$ , and let  $d$  be such that  $dN_0 \mid N$ . Then  $f|B_d$  is orthogonal to  $P_2(N)$ .*

*Proof.* It suffices to show that  $f|B_d$  is orthogonal to each of the generators of  $P_2(N)$ , so we fix a product

$$(E_1^{1,\alpha} E_1^{1,\overline{\alpha N}})|W_S^N$$

where  $\alpha$  is a primitive odd character modulo  $M \mid N$  and  $S \subset T$  is a subset of the primes  $p$  with  $v_p(N) = 1$ . Since  $W_S^N$  is self-adjoint,

$$\langle (E_1^{1,\alpha} E_1^{1,\overline{\alpha N}})|W_S^N, f|B_d \rangle = \langle E_1^{1,\alpha} E_1^{1,\overline{\alpha N}}, f|B_d|W_S^N \rangle.$$

Using Lemma 5.2 and the fact that  $f$  is an eigenfunction of all  $W_{S'}^M$  for sets  $S' \subseteq T$  of prime divisors of  $N_0$ , we see that  $f|B_d|W_S^N$  is a multiple of  $f|B_{d'}$  for some  $d' \mid d$ . Arguing as in Proposition 4.1,

$$\langle E_1^{1,\alpha} E_1^{1,\overline{\alpha N}}(\cdot, s), f|B_{d'} \rangle = \frac{\Gamma(s+1)}{d'^{s+1}(4\pi)^{s+1}} \sum_{n \geq 1} \frac{a_n \sigma_{1,1,\alpha}(d'n)}{n^{s+1}},$$

where  $a_n$  are the Fourier coefficients of  $f$  (note that  $f^c = f$  in this case), and  $\text{Re } s \gg 0$ . Let  $d' = \prod p^{e_p}$ . Then

$$\sum_{n \geq 1} \frac{a_n \sigma_{1,1,\alpha}(d'n)}{n^{s+1}} = \sum_{\gcd(n, d')=1} \frac{a_n \sigma_{1,1,\alpha}(n)}{n^{s+1}} \prod_{p \mid d'} \left( \sum_{b=0}^{\infty} \frac{a_{p^b} \sigma_{1,1,\alpha}(p^{b+e_p})}{(p^b)^{s+1}} \right) \quad (27)$$

The first sum over  $n$  coprime to  $d'$  is, up to the Euler factors corresponding to the prime divisors of  $d'$ , given in the proof of Proposition 4.1. It has analytic continuation to  $s = 0$  and vanishes there, since  $L(f, 1) = 0$ . It remains to show that the sums

$$f_p(s) = \sum_{b=0}^{\infty} \frac{a_{p^b} \sigma_{1,1,\alpha}(p^{b+e_p})}{(p^b)^{s+1}} = \sum_{b=0}^{\infty} \frac{a_{p^b} \sigma_{1,1,\alpha}(p^b)}{(p^b)^{s+1}} + \sum_{b=0}^{\infty} \frac{a_{p^b} \alpha(p^b) p^b}{(p^b)^{s+1}} (\alpha(p)p + \dots + \alpha(p^{e_p})p^{e_p})$$

can be analytically continued to  $s = 0$  for all  $p$  dividing  $d'$ . The first sum corresponds to the Euler factors at  $p$  of the quotient of  $L$ -functions given in Proposition 4.1 and hence has analytic continuation to  $s = 0$ . The second sum equals

$$(\alpha(p)p + \dots + \alpha(p^{e_p})p^{e_p}) L_p(f_\alpha, s),$$

which can again be analytically continued to  $s = 0$ .  $\square$

We now define a space  $\mathcal{S}_{2,\text{rk}=0}(N)$  that projects onto  $\mathcal{S}_{2,\text{rk}=0}^{\text{new}}(N)$  from Theorem 4.4, by

$$\mathcal{S}_{2,\text{rk}=0}(N) = \bigcup_{N_0 \mid N} \bigcup_{d \mid N/N_0} \mathcal{S}_{2,\text{rk}=0}^{\text{new}}(N_0)|B_d.$$

By Proposition 5.5  $P_2(N)$  is contained in  $\mathcal{S}_{2,\text{rk}=0}(N)$  and by Theorem 5.4 the two spaces have the same projection to the new space of  $\mathcal{S}_2(N)$ . This projection is equal to the projection of  $\mathcal{Q}_2(N)$  and so we can again use induction to prove:

**Theorem 5.6.** *Let  $N$  be as in Theorem 4.4 and let  $\mathcal{Q}_2(N) = \cup_{N_0 d \mid N} \mathcal{Q}_2(N_0)|B_d$  be the subspace of  $\mathcal{M}_2(N)$  generated by the products*

$$E_1^{\phi,\psi}|B_{d_1 d} \cdot \overline{E_1^{\phi,\psi}}|B_{d_2 d}$$

for all  $(\phi, \psi, 1, d_1, d_2) \in B'(N_0)$ . Then

$$\mathcal{S}_{2,\text{rk}=0}(N) + \mathcal{E}_2(N) = \mathcal{Q}_2(N) + \mathcal{E}_2(N).$$

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